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A CRITERION OF THE LINEAR LONG-WAVE STABILITY OF STEADY MAGNETOHYDRODYNAMIC JET FLOWS OF AN IDEAL FLUID†

Yu. G. GUBAREV and V. V. NIKULIN

Novosibirsk

e-mail: gubarev@hydro.nsc.ru

nikulin@hydro.nsc.ru

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The problem of the linear stability of steady axisymmetric shear magnetohydrodynamic jet flows of an inviscid ideally conducting incompressible fluid with a free boundary is investigated. It is assumed that the jet is of unlimited length, there is a longitudinal constant electric current along its surface, and it is directed along the axis of a cylindrical shell with infinite conductivity, such that there is a vacuum layer between its free boundary and the inner surfaces of the shell. The necessary and sufficient condition for the stability of such flows with respect to small axisymmetric long-wave perturbations of special form is obtained by Lyapunov's direct method. Bilateral exponential estimates of the growth of small perturbations are constructed in the case when this stability condition breaks down, where the indices in their exponents are calculated from the parameters of the steady flows and the initial data for the perturbations. An example of a steady axisymmetric shear magnetohydrodynamic jet flow and of the initial small axisymmetric long-wave perturbations imposed on it is given, which, at the linear stage, will evolve in time and space in accordance with the estimates constructed. © 2003 Elsevier Ltd. All rights reserved.

Below, the results on stability are established for a considerably wider class of steady flows than those considered previously in [1], and when there is a poloidal rather than an azimuthal magnetic field, "frozen" into the material of the jet.

1. FORMULATION OF THE EXACT PROBLEM

We will investigate an axisymmetric ideally conducting fluid jet of unlimited length in a magnetic field, along the free boundary of which there is a constant longitudinal electric current J. The jet is directed along the axis of a cylindrical shell of radius r_* of infinite conductivity, which nowhere touches it due to the presence of a vacuum gap between them. We will introduce a cylindrical system of coordinates (r^*, φ, z^*) such that its z^* axis coincides with the axis of symmetry of the jet. We will use the following notation: ρ is the fluid density, (v_1, v_2, v_3) are the components of the velocity field, (H_1, H_2, H_3) are the components of the magnetic field inside the jet, P is the pressure field, (H_1^*, H_2^*, H_3^*) are the components of the magnetic field outside the conducting jet, and t^* is the time. We will assume that when the fluid in the jet moves $v_2 \equiv 0, H_2 \equiv 0$, these motions are themselves axisymmetric, and the fluid is ideal, incompressible and of uniform density. The action of the surface-tension forces on the free boundary of the jet is ignored.

In the light of the above assumptions, the equations of single fluid non-dissipative magnetohydrodynamics [2, 3] take the form

$$\rho\left(\frac{\partial v_1}{\partial t^*} + v_1\frac{\partial v_1}{\partial r^*} + v_3\frac{\partial v_1}{\partial z^*}\right) = -\frac{\partial P_*}{\partial r^*} + \frac{H_1}{4\pi}\frac{\partial H_1}{\partial r^*} + \frac{H_3}{4\pi}\frac{\partial H_1}{\partial z^*}$$

$$\rho\left(\frac{\partial v_3}{\partial t^*} + v_1\frac{\partial v_3}{\partial r^*} + v_3\frac{\partial v_3}{\partial z^*}\right) = -\frac{\partial P_*}{\partial z^*} + \frac{H_1}{4\pi}\frac{\partial H_3}{\partial r^*} + \frac{H_3}{4\pi}\frac{\partial H_3}{\partial z^*}$$

$$\frac{1}{r^*}\frac{\partial (v_1 r^*)}{\partial r^*} + \frac{\partial v_3}{\partial z^*} = 0 \tag{1.1}$$

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$$\frac{\partial(Ar^*)}{\partial t^*} + v_1\frac{\partial(Ar^*)}{\partial r^*} + v_3\frac{\partial(Ar^*)}{\partial z^*} = 0$$

$$P_* \equiv P + \frac{H_1^2 + H_3^2}{8\pi}, \quad H_1 \equiv -\frac{\partial A}{\partial z^*}, \quad H_3 \equiv \frac{1}{r^*}\frac{\partial(Ar^*)}{\partial r^*}$$

Here P_* is the modified pressure field and A is the azimuthal component of the magnetic field vector potential.

If we neglect the displacement current, the relations for the components of the magnetic field in the region between the inner surface of the cylindrical shell and the free boundary of the jet [4] can be written in the form

$$\frac{\partial H_2^*}{\partial z^*} = 0, \quad \frac{\partial H_1^*}{\partial z^*} - \frac{\partial H_3^*}{\partial r^*} = 0, \quad \frac{1}{r^*}\frac{\partial(H_2^* r^*)}{\partial r^*} = 0$$
$$\frac{1}{r^*}\frac{\partial(H_1^* r^*)}{\partial r^*} + \frac{\partial H_3^*}{\partial z^*} = 0 \tag{1.2}$$

We impose the following boundary conditions:
on the axis of the jet

$$r^* = 0: v_1 = 0, \quad H_1 = 0 \tag{1.3}$$

on its free surface

$$r^* = r_1(t^*, z^*): P_* = \frac{H_1^{*2} + H_2^{*2} + H_3^{*2}}{8\pi}, \quad v_1 = \frac{\partial r_1}{\partial t^*} + v_3\frac{\partial r_1}{\partial z^*}$$
$$H_1 - H_3\frac{\partial r_1}{\partial z^*} = 0, \quad H_1^* - H_3^*\frac{\partial r_1}{\partial z^*} = 0 \tag{1.4}$$

on the inner boundary of the cylindrical shell

$$r^* = r_*: H_1^* = 0 \tag{1.5}$$

The initial conditions for system of equations (1.1) and the second of relations (1.4) are specified in the form

$$v_1(0, r^*, z^*) = v_{10}(r^*, z^*), \quad v_3(0, r^*, z^*) = v_{30}(r^*, z^*)$$
$$A(0, r^*, z^*) = A_0(r^*, z^*), \quad r_1(0, z^*) = r_{10}(z^*) \tag{1.6}$$

where it is required of the functions v_{10}, v_{30}, A_0 and r_{10} that they should not contradict the third equation of system (1.1), conditions (1.3), and also the first, third and fourth relations of system (1.4).

Further, in the initial-boundary-value problem (1.1)–(1.6) we change to the long-wave approximation, anticipating the change to dimensionless form. Here, we take the following as the scales: L is the characteristic scale of change of the hydrodynamic and magnetic fields along the z^* coordinate axis, v_0 is the characteristic velocity of the fluid, and r_0 is the characteristic radius of the jet, and we construct the dimensionless quantities $t, \eta, p_*, q, z, H, w, a, h^*, \kappa$ and H^*, so that we have the following expressions

$$t^* = tL/v_0, \quad r^{*2} = \eta L^2\delta^2, \quad P_* = p_*\rho v_0^2, \quad 2v_1 r^* = qv_0 L\delta^2$$
$$z^* = zL, \quad H_1 r^* = hv_0\sqrt{\pi\rho}L\delta^2, \quad H_3 = 2Hv_0\sqrt{\pi\rho}$$
$$v_3 = wv_0, \quad Ar^* = av_0\sqrt{\pi\rho}L^2\delta^2, \quad H_1^* r^* = h^* v_0\sqrt{\pi\rho}L\delta^2$$
$$H_2^* r^* = 2\kappa v_0\sqrt{\pi\rho}L\delta, \quad H_3^* = 2H^* v_0\sqrt{\pi\rho} \tag{1.7}$$

where $\delta \equiv r_0/L \ll 1$ is the dimensionless characteristic radius of the jet.

As a result we obtain the following system of equations from (1.1)

$$\begin{aligned} &\delta^2\left[q_t + qq_\eta - \frac{q^2}{2\eta} + wq_z\right] = -4\eta p_{*\eta} + \delta^2\left[hh_\eta - \frac{h^2}{2\eta} + Hh_z\right] \\ &w_t + qw_\eta + ww_z = -p_{*z} + hH_\eta + HH_z, \quad q_\eta + w_z = 0 \\ &a_t + qa_\eta + wa_z = 0; \quad h \equiv -a_z, \quad H \equiv a_\eta \end{aligned} \tag{1.8}$$

Here and below the corresponding partial derivatives are denoted by the subscripts of the independent variables.

The system of relations (1.2) is converted to the form

$$\delta^2 h_z^* - 4\eta H_\eta^* = 0, \quad H_z^* + h_\eta^* = 0, \quad \kappa \equiv \frac{J}{r_0 \upsilon_0 \sqrt{\pi\rho}} = \text{const} \tag{1.9}$$

Bearing the last equation of (1.9) in mind, the boundary conditions (1.3)–(1.5) can be represented in the form

$$\begin{aligned} &\eta = 0: q = 0, \quad h = 0 \\ &\eta = \eta_1(t, z): q = \eta_{1t} + w\eta_{1z}, \quad p_* = \frac{1}{2}\left[\frac{(h^*\delta)^2}{4\eta_1} + \frac{\kappa^2}{\eta_1} + H^{*2}\right] \\ &\eta = \eta_1(t, z): h - H\eta_{1z} = 0, \quad h^* - H^*\eta_{1z} = 0 \\ &\eta = \eta_*: h^* = 0 \end{aligned} \tag{1.10}$$

The function $\eta_1(t, z)$ describes the change in the form of the free surface of the jet with time, while the quantity η_* corresponds to the radius of the cylindrical shell surrounding it.

Finally, the initial conditions (1.6) take the form

$$\begin{aligned} &q(0, \eta, z) = q_0(\eta, z), \quad w(0, \eta, z) = w_0(\eta, z) \\ &a(0, \eta, z) = a_0(\eta, z), \quad \eta_1(0, z) = \eta_{10}(z) \end{aligned} \tag{1.11}$$

Hence it follows that if we eliminate from the relations of the initial-boundary-value problem (1.8)–(1.11) terms proportional to the factor δ^2, and remove the expression for the function $q(0, \eta, z)$, we thereby give it a form which also corresponds exactly to the long-wave approximation.

It is noteworthy that the system of equations (1.9) will then possess a solution which converts the sixth and seventh boundary conditions of (1.10) into identities, namely

$$h^*(t, \eta, z) = (\eta_* - \eta)H_z^*, \quad H^*(t, z) = \Phi/(\eta_* - \eta_1)$$

Here $\Phi = \Phi(t)$ is the derivative of a function of time, which, in its physical meaning, is the dimensionless axial magnetic flux through the vacuum layer between the free boundary of the jet and the inner surface of the cylindrical shell.

It is further assumed that the axial magnetic flux through the vacuum gap between the jet and the shell is fixed, i.e. $\Phi(t) \equiv \Phi_0 = \text{const}$. This assumption implies that there is no mechanical system of sources apart from the source investigated, which could cause a change in the value of this flux with time, and is in complete agreement with the relations obtained from the mixed problem (1.8)–(1.11) by changing to the long-wave approximation. As a result, the fourth equation of the system of boundary conditions (1.10) is converted into the relation

$$\eta = \eta_1(t, z): p_* = \frac{1}{2}\left[\frac{\kappa^2}{\eta_1} + \frac{\Phi_0^2}{(\eta_* - \eta_1)^2}\right] \tag{1.12}$$

The subsequent consideration of the long-wave modification of the initial-boundary-value problem (1.8)–(1.11) can be simplified considerably if we replace the Euler independent variables (t, η, z) by the mixed Euler–Lagrange variables (t', ν, z') [5]. This replacement, by analogy with that proposed previously in [1], is defined by the relations

$$t = t', \quad \eta = R(t', \nu, z'), \quad z = z'; \quad \nu \in [0, 1]$$

where it is assumed that the function R occurring in them satisfies the equation

$$q = R_{t'} + wR_{z'} \tag{1.13}$$

and the boundary conditions

$$R(t', 0, z') = 0, \quad R(t', 1, z') = \eta_1(t', z') \tag{1.14}$$

The essential feature of this replacement of the independent variables is the fact that the trajectories of the fluid particles now turn out to be numbered using the new variable ν. Moreover, it follows from expressions (1.13) and (1.14), which characterize the properties of the function $R(t', \nu, z')$, that the first and third boundary conditions of system (1.10) are automatically satisfied for the function $q(t, \eta, z)$. Finally (and this is very important), the unknown free boundary $\eta = \eta_1(t, z)$ of the jet is mapped by such a replacement of the independent variables into the known fixed surface $\nu = 1$.

Thus, in the new mixed Euler–Lagrange variables (taking into account the fact that terms containing the factor δ^2 are neglected), the system of relations (1.8) can be rewritten in the form

$$\begin{gathered} R_\nu(w_t + ww_z) = -R_\nu p_{*z} + hH_\nu + R_\nu HH_z - HR_zH_\nu \\ p_{*\nu} = 0, \quad q_\nu + R_\nu w_z - R_z w_\nu = 0, \quad a_t + wa_z = 0 \\ h \equiv -a_z + \frac{a_\nu R_z}{R_\nu}, \quad H \equiv \frac{a_\nu}{R_\nu} \end{gathered} \tag{1.15}$$

where, for the convenience in representing the calculations which follow, the primes are omitted from the independent variables t' and z'.

The boundary conditions for Eqs (1.15) will be relations (1.12) and the constraint

$$a_z = 0 \quad (\nu = 0, 1) \tag{1.16}$$

following from the second and fifth equations of (1.10), and also the last two expressions of system of equations (1.15).

Relations (1.12)–(1.16) are supplemented by the initial conditions in the form

$$w(0, \nu, z) = w_0(\nu, z), \quad R(0, \nu, z) = R_0(\nu, z), \quad a(0, \nu, z) = a_0(\nu, z) \tag{1.17}$$

The function $R_0(\nu, z)$ is assumed to be a monotonically increasing function of the argument ν in view of the requirement of the one-to-one correspondence of the product of the replacement of variables.

The mixed problem (1.12)–(1.17) will be investigated further on the assumption that the quantity a is a specified function of the independent variable ν, namely, $a = a_*(\nu)$. This means that the product of the azimuthal components of the vector potential of the magnetic field and the radial coordinate, reduced to dimensionless form, remains constant on each line ν = const during the motion. It is essential that this constraint does not contradict the fourth term from Eqs (1.15), since if we take $a_0 = a_*(\nu)$ when $t = 0$, then, according to this equation, this form of the dependence of a on the variable ν does not undergo any changes at any subsequent instant of time $t > 0$. Moreover, if the function a has the form $a_*(\nu)$, the boundary condition (1.16) is satisfied identically.

In order to give the initial-boundary-value problem (1.12)–(1.17) the clearest form, by means of relation (1.12) and the second equation of system (1.15) we obtain the relation

$$p_{*z} = \left[-\frac{\kappa^2}{2\eta_1^2} + \frac{\Phi_0^2}{(\eta_* - \eta_1)^3}\right]\eta_{1z} \tag{1.18}$$

Now replacing the partial derivative p_{*z} in the system of relations (1.15) by its expression (1.18), and the function q by its representation (1.13) and taking the above assumption on the form of the functional relation between a and the variable ν into account, we can finally write the mixed problem (1.12)–(1.17) in the form

$$
\begin{aligned}
& w_t + ww_z = \left[\frac{\kappa^2}{2R_1^2} - \frac{\Phi_0^2}{(R_* - R_1)^3}\right] R_{1z} - \frac{R_{\nu z}}{R_\nu^3}\left(\frac{da_*}{d\nu}\right)^2 \\
& R_{\nu t} + (wR_\nu)_z = 0 \\
& h \equiv \frac{R_z}{R_\nu}\frac{da_*}{d\nu}, \quad H \equiv R_\nu^{-1}\frac{da_*}{d\nu} \\
& w(0, \nu, z) = w_0(\nu, z), \quad R(0, \nu, z) = R_0(\nu, z)
\end{aligned}
\tag{1.19}
$$

where R_1 is the value of the function R on the free boundary of the jet $\nu = 1$, for which, by virtue of the second of equations (1.14), we have the relation $R_1 \equiv \eta_1(t, z)$, while R_* is the radius of the cylindrical shell surrounding the jet (this notation is introduced instead of η_* for the sake of uniformity of the further discussion).

It is necessary to note that, if we choose as R a function which decreases monotonically with respect to the independent variable ν, in this case we obtain equations similar to the first two relations of system (1.19). The difference will consist solely of the fact that the free surface of the jet will then not be the axis $\nu = 1$ but the straight line $\nu = 0$, while its axis of symmetry will be the straight line $\nu = 1$ and not the axis $\nu = 0$.

An energy integral exists for the initial-boundary-value problem (1.19) of the form

$$
E_1 \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\left(\int_0^1\left[w^2R_\nu + R_\nu^{-1}\left(\frac{da_*}{d\nu}\right)^2\right]d\nu + \kappa^2\ln R_1 + \frac{\Phi_0^2}{R_* - R_1}\right)dz = \text{const} \tag{1.20}
$$

with the condition that the solutions of this problem are either periodic along the coordinate z axis or localized on it (in the latter case the flow of fluid at infinity must be uniform along the z coordinate). Moreover, it is easy to show that the functional

$$
I \equiv \int_{-\infty}^{+\infty}\int_0^1 [w_\nu F(a_*)]d\nu dz \tag{1.21}
$$

where F is an arbitrary function, serves as one more integral of motion for the mixed problem (1.19)[1.6].

For later convenience we will introduce the following notation

$$
\chi_{mn} = \chi_{mn}(\nu) \equiv \left(\frac{dR^0}{d\nu}\right)^{-m}\left(\frac{da_*}{d\nu}\right)^n
$$

The initial-boundary-value problem (1.19) has exact steady solutions, which can be written in the form

$$
w = w^0(\nu), \quad R = R^0(\nu), \quad R_1 = R_1^0 \equiv 1, \quad h = h^0 \equiv 0, \quad H = H^0(\nu) \equiv \chi_{11} \tag{1.22}
$$

where w^0 is an arbitrary function of the variable ν while R^0 is a monotonically increasing function of the variable ν, and the radius of the unperturbed jet is taken to be equal to r_0 (1.7).

Our further investigation is aimed at clarifying the sufficient conditions for linear stability of the stationary solutions (1.22) to small axisymmetric long-wave perturbations $w'(t, \nu, z)$ and $R'(t, \nu, z)$.

2. FORMULATION OF THE LINEARIZED PROBLEM

We can attempt to achieve the above-mentioned aim by linearizing the mixed problem (1.19) near the exact stationary solutions (1.22). We finally obtain the initial-boundary-value problem

$$
\begin{aligned}
& w'_t + w^0 w'_z = -\Psi R'_{1z} - \chi_{32}R'_{\nu z}, \quad R'_{\nu t} + w^0 R'_{\nu z} + \frac{dR^0}{d\nu}w'_z = 0 \\
& R'_1(t, z) \equiv R'(t, 1, z), \quad h' \equiv \chi_{11}R'_z, \quad H' \equiv -\chi_{21}R'_\nu \\
& w'(0, \nu, z) = w'_0(\nu, z), \quad R'(0, \nu, z) = R'_0(\nu, z)
\end{aligned}
\tag{2.1}
$$

Here

$$\Psi \equiv \frac{\Phi_0^2}{(R_* - 1)^3} - \frac{\kappa^2}{2}$$

In the solutions of this problem the functional

$$E \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\left(\int_0^1\left[\frac{dR^0}{d\nu}w'^2 + 2w^0R'_\nu w' + \chi_{32}R_\nu'^2\right]d\nu + \Psi R_1'^2\right)dz \tag{2.2}$$

remains the same with time.

It is easy to check that the first variation ∂J_1 of the integral $J_1 \equiv E_1 + I$ (1.20), (1.21) vanishes in the stationary solutions (1.22), if the functions w^0, R^0, a_* and F convert the equations

$$w^0\frac{dR^0}{d\nu} = \frac{dF}{da_*}\frac{da_*}{d\nu}, \quad \chi_{11}\frac{d\chi_{11}}{d\nu} = w^0\frac{dw^0}{d\nu} \tag{2.3}$$

into identities. Here the second variation $\delta^2 J_1$ of functional J_1, rewritten in appropriate notation, is identical in form with the integral E (2.2).

The exact stationary solutions (1.22) of mixed problem (1.19) will be stable to small axisymmetric long-wave perturbations (2.1) if and only if the functional E is sign-definite.

In order to establish whether the integral E (2.2) possesses the property of sign-definiteness, it is convenient to represent it in the form

$$E = \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1 (B\mathbf{u}, \mathbf{u})d\nu dz; \quad \mathbf{u} \equiv \mathrm{col}(w', R'_\nu, R'_1) \tag{2.4}$$

Here $B = \|b_{ik}\|$ is a 3×3 matrix with non-zero elements

$$b_{11} = \frac{dR^0}{d\nu}, \quad b_{12} = b_{21} = w^0, \quad b_{22} = \chi_{32}, \quad b_{23} = b_{32} = \frac{\Psi}{2}$$

By Sylvester's criterion [7] the integrand of the functional E (2.4) is positive (negative) definite if and only if the principal minors of the matrix B are positive (have the sign $(-1)^n$, when n is the order of the corresponding principal minor). Direct calculation of the principal minor Δ_n of the matrix B leads to the relations

$$\Delta_1 = \frac{dR^0}{d\nu} > 0, \quad \Delta_2 = \chi_{22} - w^{02}, \quad \Delta_3 = -\frac{\Psi^2}{4}\frac{dR^0}{d\nu} < 0$$

and enables us to conclude that, by virtue of Sylvester's criterion, the integral E (2.4) does not possess the property of sign-definiteness.

Hence, one cannot obtain the sufficient conditions for linear stability of the exact stationary solutions (1.22) of the initial-boundary-value problem (1.19) to small axisymmetric long-wave perturbations $w'(t, \nu, z)$ and $R'(t, \nu, z)$ (2.1), if we mean by these the conditions for the functional E, given by (2.2) and (2.4), to be sign-definite as far as its energy nature is concerned.

3. FORMULATION OF THE SPECIAL EXACT PROBLEM

Below, we consider the linear stability of the stationary solutions (1.22) of mixed problem (1.19) in the class of motions of the fluid such that

$$R(t, \nu, z) \equiv R_1(t, z)R^0(\nu) \tag{3.1}$$

This relation enables to convert the second of the equations of the initial-boundary-value problem (1.19) to the form

$$R_{1t} + (wR_1)_z = 0 \tag{3.2}$$

If we differentiate relation (3.2) with respect to the independent variable ν, we obtain the equation

$$u_{z\nu} = 0; \quad u \equiv wR_1 \tag{3.3}$$

which is the condition for the motions of the fluid of the form (3.1) to be consistent.

Substituting the functions R from (3.1) and u from (3.3) into the first, third and fourth relations of mixed problem (1.19), and also into Eq. (3.2), we can write them in the form

$$\begin{gathered} u_t + \left(\frac{u^2}{R_1}\right)_z = \left[\frac{\kappa^2}{2R_1} - \frac{\Phi_0^2 R_1}{(R_* - R_1)^3} - \frac{\chi_{22}}{R_1^2}\right] R_{1z}, \quad R_{1t} + u_z = 0 \\ h = \frac{R^0}{R_1}\chi_{11} R_{1z}, \quad H = \frac{\chi_{11}}{R_1} \end{gathered} \tag{3.4}$$

In turn, differentiating the first of relations (3.4) first with respect to the variable ν and then with respect to the independent variable z and, finally, once again with respect to the variable ν, we derive the second condition for the class (3.1) of motions of the fluid to be compatible

$$\left[\left(\frac{da_*}{d\nu}\frac{d\chi_{11}}{d\nu}\right)^{-1}\frac{dR^0}{d\nu}\left(\frac{u^2}{R_1}\right)_{zz\nu}\right]_\nu = 0 \tag{3.5}$$

We supplement Eqs (3.3)–(3.5) by the initial conditions

$$u(0, \nu, z) = u_0(\nu, z), \quad R_1(0, z) = R_{10}(z) \tag{3.6}$$

Finally, we formulate initial-boundary-value problem (3.3)–(3.6), which describes the special class (3.1), (3.3) and (3.5) of transient axisymmetric shear magnetohydro-dynamic jet flows of an inviscid ideally conducting incompressible fluid with a free boundary in the long-wave approximation.

The integrals E_1 (1.20) and I (1.21) will also be preserved in time in the solutions of the mixed problem (3.3)–(3.36), but, in view of the definitions of the functions R (3.1) and u (3.3) introduced above, their form becomes somewhat different:

$$\begin{gathered} E_1 = \frac{1}{2}\int_{-\infty}^{+\infty}\left(\int_0^1\left[R_1^{-1}\frac{dR^0}{d\nu}(u^2 + \chi_{22})\right]d\nu + \kappa^2 \ln R_1 + \frac{\Phi_0^2}{R_* - R_1}\right)dz \\ I = \int_{-\infty}^{+\infty}\int_0^1\left[\frac{u_\nu}{R_1}F(a_*)\right]d\nu dz \end{gathered} \tag{3.7}$$

The following functions serve as the exact stationary solutions of initial-boundary-value problem (3.3)–(3.6)

$$u = w^0(\nu), \quad R_1 = R_1^0 \equiv 1, \quad h = h^0 \equiv 0, \quad H = H^0(\nu) \equiv \chi_{11} \tag{3.8}$$

The purpose of the further investigation is to find the sufficient conditions for linear stability of stationary solutions (3.8) to small axisymmetric long-wave perturbations $u'(t, \nu, z)$ and $R_1'(t, z)$. It is clear that these stability conditions will simultaneously be the sufficient conditions for linear stability of the exact stationary solutions (1.22) of mixed problem (1.19) with respect to the small axisymmetric long-wave perturbations.

4. FORMULATION OF THE SPECIAL LINEARIZED PROBLEM

To achieve the stated aim, we will linearize initial-boundary-value problem (3.3)–(3.6) near the stationary solutions (3.8), which leads to the mixed problem

$$u'_t + 2w^0 u'_z = (w^{02} - \chi) R'_{1z}, \quad R'_{1t} + u'_z = 0, \quad u'_{z\nu} = 0$$
$$\frac{d}{d\nu}\left[w^0 - \chi_{11}\frac{d\chi_{11}}{d\nu}\left(\frac{dw^0}{d\nu}\right)^{-1}\right] = 0, \quad h' = R^0 R'_{1z}\chi_{11} \tag{4.1}$$
$$H' = -R'_1\chi_{11}; \quad u'(0,\nu,z) = u'_0(\nu,z), \quad R'_1(0,z) = R'_{10}(z)$$

Here

$$\chi \equiv \Psi + \chi_{22}$$

For this problem we have the following functional, which preserves its form with time

$$E_2 \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1\left(\frac{dR^0}{d\nu}[u'^2 + (\chi - w^{02})R_1'^2]\right)d\nu dz \tag{4.2}$$

The first variation δJ_1 of the integral $J_1 = E_1 + I$ (3.7) is equal to zero in the exact stationary solutions (3.8), while its second variation $\delta^2 J_1$ is identical, in the corresponding notation, to the functional E_2, if the following relations hold

$$w^0\frac{dR^0}{d\nu} = \frac{dF}{da_*}\frac{da_*}{d\nu}, \quad w^0(1)F(a_*(1)) = w^0(0)F(a_*(0)) \tag{4.3}$$

(compare with Eqs (2.3)).

The stationary solutions (3.8) of initial-boundary-value problem (3.3)–(3.6) (and of course, also the accurate stationary solutions (1.22) of mixed problem (1.19)) will be stable to small axisymmetric long-wave perturbations (4.1) if and only if everywhere inside the jet

$$\chi - w^{02} \geq 0 \tag{4.4}$$

since this relation, as follows from expression (4.2), ensures at least that the integral E_2 is negative.

We must emphasise that inequality (4.4) is also the sufficient condition for the linear stability of the stationary solutions (3.8) (or (1.22)), which it was required to obtain.

5. THE LYAPUNOV FUNCTIONAL

We will further assume that relation (4.4) remains true at least everywhere inside the jet. In this case we can hope to demonstrate the linear instability of the accurate stationary solutions (3.8) of initial-boundary-value problem (3.3)–(3.6) (and together with these, naturally, the stationary solutions (1.22) of mixed problem (1.19) also) to small axisymmetric long-wave perturbations $u'(t,\nu,z)$ and $R_1'(t,z)$ (4.1). If this can be achieved, then we will have proved that condition (4.4) of linear stability of the exact stationary solutions (3.8) (or (1.22)) is not only sufficient but also necessary.

To do this we must be able to separate from the small axisymmetric long-waveperturbation (4.1) at most one, but which increases with time exponentially rapidly. This can be most effectively realised in the case when the investigation is concentrated on small axisymmetric long-wave perturbations which are simple deviations of the trajectories of motion of the fluid particles from the corresponding streamlines of the steady flows (3.8). These perturbations can be described most clearly of all using the Lagrange displacement field $\xi = \xi(t,\nu,z)$ [8], which satisfies the equation

$$\xi_t = u' \tag{5.1}$$

Hence it follows that initial-boundary-value problem (4.1) then takes the form

$$\xi_{tt} + 2w^0\xi_{tz} = (\chi - w^{02})\xi_{zz}, \quad R'_1 = -\xi_z, \quad \xi_{z\nu} = 0, \quad H' = \chi_{11}\xi_z$$
$$\frac{d}{d\nu}\left[w^0 - \chi_{11}\frac{d\chi_{11}}{d\nu}\left(\frac{dw^0}{d\nu}\right)^{-1}\right] = 0, \quad h' = -R^0\chi_{11}\xi_{zz} \tag{5.2}$$
$$\xi(0,\nu,z) = \xi_0(\nu,z), \quad \xi_t(0,\nu,z) = u'(0,\nu,z) = u'_0(\nu,z)$$

Due to the presence of the third relation in system (5.2), the mixed problem (5.1), (5.2) is overdefined. However, this relation indicates that the function $\xi(t, \nu, z)$, being a solution of initial-boundary-value problem (5.1), (5.2), must have the form

$$\xi(t, \nu, z) = f(t, z) + f_1(t, \nu) \tag{5.3}$$

($f(t, z)$ and $f_1(t, \nu)$ are certain functions of its arguments) and no other.

Consequently, the initial data $\xi_0(\nu, z)$ and $u_0'(\nu, z)$ for mixed problem (5.1), (5.2) must be specified in the form

$$\xi_0(\nu, z) = f_2(\nu) + f_3(z), \quad u_0'(\nu, z) = f_4(\nu) + f_5(z) \tag{5.4}$$

where f_2 and f_4 are arbitrary functions of the independent variable ν, and f_3 and f_5 are certain functions of the variable z.

We must now investigate whether initial-boundary-value problem (5.1), (5.2) has solutions in the form (5.3) and whether the fact that mixed problem (5.1), (5.2) is overdefined imposes any additional limitations (apart from (5.4)) on the choice of the initial perturbations $\xi_0(\nu, z)$, $u_0'(\nu, z)$.

The most convincing answers to these questions can be given if we reformulate initial-boundary-value problem (5.1), (5.2) in the form

$$\begin{gathered} R'_{1tt} + 2w^0 R'_{1tz} = (\chi - w^{02}) R'_{1zz}, \quad h' = R^0 \chi_{11} R'_{1z} \\ \frac{d}{d\nu}\left[w^0 - \chi_{11}\frac{d\chi_{11}}{d\nu}\left(\frac{dw^0}{d\nu}\right)^{-1}\right] = 0, \quad H' = -\chi_{11} R'_1 \\ R'_1(0, z) = R'_{10}(z)\left(= -\frac{df_3}{dz}\right), \quad R'_{1t}(0, z) = (R'_{1t})_0(z)\left(= -\frac{df_5}{dz}\right) \end{gathered} \tag{5.5}$$

It can be seen that mixed problem (5.5) includes a unique equation for determining the unique required function $R_1'(t, z)$, and, which is extremely important, this question is homogeneous. Hence, the initial data $R_{10}'(z)$ and $(R_{1t}')_0(z)$ can take arbitrary values without any additional limitations.

Finally, if the solution $R_1'(t, z)$ of initial-boundary-value problem (5.5) is obtained, then, using the second of the system of relations (5.2), one can calculate the function $\xi(t, \nu, z)$ as the solution of mixed problem (5.1), (5.2), which corresponds completely to representation (5.3).

Thus, basing ourselves on these considerations, we can conclude that initial-boundary-value problem (5.1), (5.2) can have a solution in the form (5.3), where the fact that it is overdefined is not accompanied by any additional limitations (with the exception of (5.4)) on the initial perturbations $\xi_0(\nu, z)$, $u_0'(\nu, z)$.

Below, for later convenience, we will introduce the additional integrals

$$\begin{gathered} M \equiv \int_{-\infty}^{+\infty}\int_0^1 \frac{dR^0}{d\nu}\xi^2 d\nu dz, \quad T \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1 \frac{dR^0}{d\nu}(u' - w^0 R_1')^2 d\nu dz \\ \Pi \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1 \frac{dR^0}{d\nu}\chi R_1'^2 d\nu dz, \quad \Pi_1 \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1 \frac{dR^0}{d\nu}(\chi - w^{02}) R_1'^2 d\nu dz \\ T_1 \equiv \int_{-\infty}^{+\infty}\int_0^1 \frac{dR^0}{d\nu} w^0 (u' - w^0 R_1') R_1' d\nu dz, \quad T_2 \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1 \frac{dR^0}{d\nu} u'^2 d\nu dz \end{gathered} \tag{5.6}$$

so that

$$E_2 \equiv T + \Pi + T_1 \equiv T_2 + \Pi_1 = \text{const} \tag{5.7}$$

Double differentiation of the function M (5.6) with respect to time and conversions of the integral obtained in the result using relations (4.1), (5.1), (5.2) and (5.6) enables us to arrive at the fundamental equation [1.9]

$$d^2M/dt^2 = 4(T - \Pi)$$

which is called a virial equality [8]. Now multiplying this equality by a certain constant factor λ, bearing relation (5.7) in mind and using the equation

$$dT_1/dt = 0$$

(which follows directly from the mixed problems (4.1) and (5.1), (5.2)), we derive the key relation

$$dE_\lambda/dt = 2\lambda E_\lambda - 4\lambda T_\lambda \tag{5.8}$$

where

$$E_\lambda \equiv T_\lambda + \Pi_\lambda, \quad 2\Pi_\lambda \equiv 2\Pi + \lambda^2 M$$

$$2T_\lambda \equiv 2T + \lambda^2 M - \lambda\frac{dM}{dt} = \int_{-\infty}^{+\infty}\int_0^1 \frac{dR^0}{d\nu}(u' - w^0 R_1' - \lambda\xi)^2 d\nu dz \geq 0$$

Choosing the factor λ to be strictly positive, we reduce Eq. (5.8), taking into account the fact that the functional T_λ is non-negative, to the differential inequality

$$dE_\lambda/dt \leq 2\lambda E_\lambda$$

integration of which gives the important limit

$$E_\lambda(t) \leq E_\lambda(0)\exp(2\lambda t) \tag{5.9}$$

It is noteworthy that inequality (5.9) is true both for any solution of initial-boundary-value problem (5.1), (5.2), and for arbitrary positive values of the constant quantity λ. Moreover (which is extremely important), when obtaining this inequality no limitations were imposed on the sign of the function Π.

Finally, relation (5.9) enables us to conclude that the integral E_λ, in a global sense, changes monotonically with time. This serves as a basis for taking it as the Lyapunov functional below [1, 9, 10].

6. BILATERAL EXPONENTIAL ESTIMATES OF THE GROWTH OF PERTURBATIONS

We will now construct a priori bilateral exponential estimates of the growth of small axisymmetric long-wave perturbations (5.1)–(5.4) of stationary solutions (3.8) of mixed problem (3.3)–(3.6) using the fundamental integral inequality (5.9), and also by an appropriate choice of the functions $\xi_0(\nu, z)$ and $u_0'(\nu, z)$ when the following inequality holds either everywhere inside the jet or only in a certain part of it

$$\chi - w^{02} < 0 \tag{6.1}$$

In this case we can take the initial perturbations $\xi_0(\nu, z)$ and $u_0'(\nu, z)$ (5.2) and (5.4) to be such that the following relations are satisfied

$$\Pi_1(0) < 0, \quad T_2(0) - T_1(0) < |\Pi_1(0)| \tag{6.2}$$

The functional $E_\lambda(0)$ (5.8) then becomes a second-degree polynomial in the parameter λ with a positive coefficient $M(0)$ (5.6) for λ^2 and a negative free term $E_3(0) \equiv E_2(0) - T_1(0)$ (4.2), (5.6), (5.7)

$$E_\lambda(0) = E_3(0) - \frac{\lambda}{2}\frac{dM}{dt}(0) + \lambda^2 M(0) \tag{6.3}$$

If we assume that

$$0 < \lambda < \Lambda \equiv A_1 + \sqrt{A_2} \tag{6.4}$$

$$A_1 \equiv \frac{1}{4M(0)}\frac{dM}{dt}(0), \quad A_2 \equiv A_1^2 - \frac{E_3(0)}{M(0)}$$

then, using expression (6.3), the integral $E_\lambda(0)$ can be given an upper limit, i.e.

$$E_\lambda(0) < 0 \tag{6.5}$$

Inequalities (5.9) and (6.5) obviously confirm the exponential growth of the small axisymmetric long-wave perturbation (5.1)–(5.4) with time.

Assuming $\lambda \equiv \Lambda - \delta_1$ (with any constant quantity $\delta_1 \in]0, \Lambda[$), relation (5.9) can be represented in the form

$$E_{\Lambda-\delta_1}(t) \le E_{\Lambda-\delta_1}(0)\exp[2(\Lambda-\delta_1)t] \quad (E_{\Lambda-\delta_1}(0) < 0) \tag{6.6}$$

Since, in accordance with the definitions (5.8) of the functions E_λ, T_λ and Π_λ, and also with expression (5.6) for the integral Π_1, the relation

$$E_\lambda(t) > \Pi_1(t)$$

is true, inequality (6.6) can be rewritten in the more indicative form

$$-\Pi_1(t) > |E_{\Lambda-\delta_1}(0)|\exp[2(\Lambda-\delta_1)t]$$

or

$$\int\limits_{-\infty}^{+\infty}\int\limits_0^1 \frac{dR^0}{d\nu}(w^{02}-\chi)R_1'^2 d\nu dz > 2|E_{\Lambda-\delta_1}(0)|\exp[2(\Lambda-\delta_1)t] \tag{6.7}$$

Relation (6.7) clearly demonstrates that the parameter $\Lambda - \delta_1$ (6.4), (6.6) is the lower limit of the permissible values of the increments of the small axisymmetric long-wave perturbation (5.1)–(5.4) of the exact stationary solutions (3.8) of initial-boundary-value problem (3.3)–(3.6).

Suppose the following inequality is true

$$\lambda > \Lambda^+ \equiv \sup_{\xi_0(\nu,z),\, u_0'(\nu,z)} \Lambda \tag{6.8}$$

In this case the functional $E_\lambda(0)$ will be positive-definite for any possible initial data $\xi_0(\nu, z)$ and $u_0'(\nu, z)$ (5.2), (5.4).

Finally, assuming $\lambda \equiv \Lambda^+ + \delta_2$ (δ_2 is an arbitrary positive constant), we convert relation (5.9) into the inequality

$$E_{\Lambda^+ + \delta_2}(t) \le E_{\Lambda^+ + \delta_2}(0)\exp[2(\Lambda^+ + \delta_2)t] \tag{6.9}$$

which means that the quantity $\Lambda^+ + \delta_2$ is an upper estimate of the value of the increments of the small axisymmetric long-wave perturbations (5.1)–(5.4) of the stationary solutions (3.8) of mixed problem (3.3)–(3.6).

Comparing relations (6.7) and (6.9) we can conclude that for a rate of growth ω of the small axisymmetric long-wave perturbations (5.1)–(5.4), the parameter Λ^+ (6.4), (6.8) serves to construct both upper and lower boundaries

$$\Lambda^+ - \delta_1 \le \omega \le \Lambda^+ + \delta_2 \tag{6.10}$$

The double inequality (6.10) indicates that those small axisymmetric long-wave perturbations (5.1)–(5.4) increase most rapidly of all, the increments of which are close in value to the value of the parameter Λ^+.

Thus, if relation (6.1) holds, then after calculating the value of the parameter Λ^+, using expressions (6.4) and (6.8), which characterizes the rate of increase ω (6.10) of the most rapidly growing small axisymmetric long-wave perturbations (5.1)–(5.4), we can answer the following question: after what time will small axisymmetric long-wave perturbations (5.1)–(5.4) lead to the destruction of stationary axisymmetric shear magnetohydrodynamic jet flow (3.8) (or, which is equivalent, (1.22)) of an inviscid ideally conducting incompressible fluid of uniform density with a free surface?

Note that the results discussed in this section are a proof of the fact that condition (4.4) for linear stability of the exact stationary solutions (3.8) (or (1.22)) is simultaneously also sufficient and necessary.

From the mathematical point of view, the results presented in this paper are a priori, since theorems of the existence of solutions of the mixed problems considered for systems of partial differential equations have not been proved.

7. EXAMPLES

We will investigate the steady axisymmetric shear magnetohydrodynamic jet flow

$$\begin{gathered} w^0(\nu) = 2-\nu, \quad R^0(\nu) = \nu, \quad R_1^0 = 1, \quad h^0 = 0, \quad H^0(\nu) = \nu+2 \\ a_*(\nu) = \nu^2/2+2\nu+1, \quad \kappa = 2, \quad \Phi_0 = \sqrt{2}, \quad R_* = 2 \end{gathered} \tag{7.1}$$

of an ideal incompressible fluid of uniform density with infinite conductivity in a region which can be imagined as an unbounded strip

$$[(z, \nu): -\infty < z < +\infty, 0 \le \nu \le 1] \tag{7.2}$$

It is clear that this flow is a typical representative of flows corresponding to the stationary solutions (1.22) of initial-boundary-value problem (1.19) (and of course also corresponding to the exact stationary solutions (3.8) of mixed problem (3.3)–(3.6)).

For the flow (7.1), (7.2) the function $F(a_*)$ (1.21) must satisfy relations (4.3), which, in the situation considered, take the form

$$dF/d\nu = 2-\nu, \quad 2F(0) = F(1)$$

whence it follows that

$$F(\nu) = -\nu^2/2+2\nu+3/2$$

Further, the flow (7.1), (7.2) is such that inequality (4.4) is satisfied for it. Really, direct calculations show that

$$\chi - w^{02} = 8\nu \ge 0$$

over the whole range (7.2) of variation of the independent variable ν.

This indicates that the flow (7.1), (7.2) will be stable in the linear approximation to small axisymmetric long-wave perturbations (4.1) and all the more (5.1)–(5.4).

We will now investigate the steady axisymmetric shear magnetohydrodynamic jet flow

$$\begin{gathered} w^0(\nu) = 4-\nu, \quad R^0(\nu) = \nu, \quad R_1^0 = 1, \quad h^0 = 0 \\ a_*(\nu) = \frac{\nu+1}{2}N(\nu)+\frac{1}{2}\ln|\nu+1+N(\nu)|+1, \quad \kappa = 8 \\ H^0(\nu) = N(\nu) \equiv \sqrt{\nu^2+2\nu+2}, \quad \Phi_0 = 4, \quad R_* = 3 \end{gathered} \tag{7.3}$$

of an inviscid ideally conducting incompressible fluid of uniform density within the same infinite strip (7.2).

This flow is also one of the representatives of flows corresponding to the stationary solutions (1.22) (and together with them, naturally, also the exact stationary solutions (3.8)). For this we will represent the function $F(a_*)$ in the form

$$F(\nu) = -\nu^2/2+4\nu+21/2$$

as a result of which the relations

$$dF/d\nu = 4-\nu, \quad 4F(0) = 3F(1)$$

are converted into identities, which follow, in the case considered, from relations (4.3).

Moreover, for the flow (7.2), (7.3)

$$\chi - w^{02} = 2(5\nu - 22)$$

i.e. inequality (6.1) is satisfied for any $\nu \in [0, 1]$.

Finally, the flow (7.2), (7.3) will be unstable, for example, to small axisymmetric long-wave perturbations (5.1)–(5.4) with initial data $\xi_0(\nu, z)$ and $u_0'(\nu, z)$ in the form

$$\xi_0(\nu, z) = \sin\frac{2\pi z}{l} + \nu, \quad u_0'(\nu, z) = 0 \tag{7.4}$$

where l is a certain positive constant quantity.

In fact, using relations (5.2), (5.6), (5.7), (6.2) and (6.3), and also taking into account the periodicity of the function $\xi_0(\nu, z)$ (7.2) with respect to the independent variable z, it is easy to establish that

$$\Pi_1(0) = \frac{4\pi^2}{l^2}\int_0^l\int_0^1\left[(5\nu-22)\cos^2\frac{2\pi z}{l}\right]d\nu dz = -\frac{39\pi^2}{l} < 0$$

$$T_2(0) - T_1(0) + \Pi_1(0) = \frac{4\pi^2}{l^2}\int_0^l\int_0^1\left[(\nu^2 - 3\nu - 6)\cos^2\frac{2\pi z}{l}\right]d\nu dz = -\frac{86\pi^2}{3l} < 0$$

whence the correctness of inequalities (6.2) follows. This in turn enables us to conclude that the flow (7.2), (7.3) is unstable to small axisymmetric long-wave perturbations (5.1)–(5.4), (7.4); they will develop with time in accordance with the limits (6.7) and (6.9) (except that in the latter we must replace Λ^+ (6.4), (6.8) by Λ (6.4)), whereas their rate of growth ω (6.10) will be determined solely by Λ.

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